Is the free locally convex space $L(X)$ nuclear?

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This is a joint work with Vladimir V. Uspenskij (Ohio University, Athens, USA). Given a class $\mathcal{P}$ of Banach spaces, a locally convex space (LCS) E is called *multi-$\mathcal{P}$* if E can be isomorphically embedded into a product of spaces that belong to $\mathcal{P}$. We investigate the question whether the free locally convex space $L(X)$ is nuclear, Schwartz, multi-Hilbert or multi-reflexive.

The following result was obtained in [1]: for every Tychonoff space X the free LCS $L(X)$ can be isomorphically embedded in the product of Banach spaces of the form $\ell^1(\Gamma)$, in other words, $L(X)$ is multi-$\mathcal{L}^1$, where $\mathcal{L}^1$ is the class of all Banach spaces of the form $\ell^1(\Gamma)$.

Addressing the question raised by Vladimir Pestov, in paper [2] we prove
Theorem 1. Let X be a k-space. The following conditions are equivalent:
(i) $L(X)$ is nuclear;
(ii) $L(X)$ is multi-Hilbert;
(iii) X is a countable discrete space.

We say that a Tychonoff space X is *projectively countable* if every metrizable image of X under a continuous map is countable.
Theorem 2. If X is a projectively countable P-space (in particular, a Lindelöf P-space), then $L(X)$ is nuclear, hence multi-Hilbert.

Addressing the question raised by Michael Megrelishvili, in paper [2] we prove
Theorem 3. Let X be a first-countable paracompact space. The following conditions are equivalent:
(i) $L(X)$ is a multi-reflexive LCS;
(ii) $L(X)$ is a Schwartz LCS;
(iii) X is locally compact and σ-compact.

In particular, Theorem 3 applies if X is metrizable.

References

[1] V. V. Uspenskij, *Unitary representability of free abelian topological groups*, Applied General Topology **9** (2008), No. 2, 197–204. https://arxiv.org/abs/math/0604253

[2] A. Leiderman, V. V. Uspenskij, *Is the free locally convex space $L(X)$ nuclear?*, (2021), http://arxiv.org/abs/2106.13413